The Logic of UNNS: On the Nature of Structure and the UNNS Topos

Abstract

In classical logic, a structure is a set equipped with interpretations of symbols from a formal language. In the Unbounded Nested Number Sequences (UNNS) Substrate , "structure" is a recursive, dynamic, and emergent notion: an attractor of operators, a stability basin of echoes, and a resonance field of recursion. We formalize UNNS structures, compare them to classical structures, and show that they form a categorical environment analogous to a topos. This situates UNNS not only as a recursive number theory but as a foundation for logic and computation.

1 Classical Notion of Structure

Definition 1 (Classical Structure)**.** *Given a language $\mathcal{L}$, a structure $\mathcal{M}$ for $\mathcal{L}$ consists of:*

- *a domain $M \neq \emptyset$,*
- *interpretations of function symbols f as maps $M^k \to M$,*
- *interpretations of relation symbols R as subsets of M^k,*
- *interpretations of constant symbols c as elements of M.*

This is a static view: M and its interpretations are fixed.

2 UNNS Definition of Structure

Definition 2 (UNNS Structure)**.** *A UNNS-structure is a quadruple*

$$\mathcal{S} = (A, \mathcal{O}, \mathcal{N}, \mathcal{R})$$

where:

- *A: base alphabet of seeds,*
- *$\mathcal{O}$: UNNS operators (Collapse, Inlaying, Inletting, Normalize, Evaluate, Adopt),*
- *$\mathcal{N}$: nesting-depth function,*
- *$\mathcal{R}$: resonance/stability map.*

Lemma 1 (Existence of Structure)**.** *A UNNS-structure exists if $(A, \mathcal{O}, \mathcal{N}, \mathcal{R})$ admits a non-trivial recursive attractor.*

3 Dynamics of Structure

- Divergence: instability under recursion $\Rightarrow$ fragility marker.
- Stabilization: hidden attractors revealed by normalization/repair.
- Resonance: operator alignment $\Rightarrow$ spectral invariants.

Proposition 1. *Sequences with bounded resonance under Collapse+Inlaying yield algebraic lattices (e.g.* $\mathbb{Z}[i], \mathbb{Z}[\omega]$*).*

4 UNNS Structures as Computation

Theorem 1 (Computability in UNNS)**.** *Every UNNS-structure simulates a recursive process. Stabilization corresponds to halting; divergence corresponds to non-halting computation.*

5 Categorical View of UNNS

5.1 Morphisms

Definition 3 (UNNS Morphism)**.** *Let* $\mathcal{S}_1 = (A_1, \mathcal{O}_1, \mathcal{N}_1, \mathcal{R}_1)$ *and* $\mathcal{S}_2 = (A_2, \mathcal{O}_2, \mathcal{N}_2, \mathcal{R}_2)$*. A morphism* $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ *is a map preserving recursion:*

$$\phi \circ O_1 = O_2 \circ \phi \quad \text{for all } O_1 \in \mathcal{O}_1,$$

and ϕ *maps attractors of* $\mathcal{S}_1$ *to attractors of* $\mathcal{S}_2$*.*

Lemma 2. *UNNS structures and morphisms form a category* **UNNS***.*

5.2 The UNNS Topos

Theorem 2 (UNNS Topos)**.** *The category* **UNNS** *has:*

1. *Products (pairing recursive structures).*
2. *Exponentials (operator substitution acts as internal hom).*
3. *A subobject classifier (collapse operator as logical truth-value object).*

Hence **UNNS** *behaves as a topos-like environment: a universe of recursive logics.*

6 Worked Example: a UNNS Topos Morphism (Fibonacci → Gaussian inlaying)

We now present a concrete example of a morphism between UNNS structures, illustrating the notion of an operator-preserving (up to controlled error) map in the category **UNNS**. The example is deliberately simple: one structure generates the classical Fibonacci recursion and includes a small collapse operator; the target structure embeds values into the Gaussian lattice via an inlaying operator. We construct an explicit map ϕ and prove commutation of the main recursion operator with the inlaying operator up to a bounded rounding error. Finally we describe how attractors (growth by the golden ratio) are transported under ϕ.

Setup

Definition 4 (Source structure $\mathcal{S}_1$). *Let $\mathcal{S}_1 = (A_1, \mathcal{O}_1, \mathcal{N}_1, \mathcal{R}_1)$ be the UNNS-structure with:*

- $A_1 = \{0, 1\}$ *(seed values for the recursion),*
- *Primary recursion operator $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ given by*

$$F(a, n) \;=\; a[n-1] + a[n-2] \quad \textit{(Fibonacci recurrence).}$$

- *Collapse operator $C_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by*

$$C_\varepsilon(x) = \begin{cases} 0, & |x| < \varepsilon, \\ x, & |x| \geq \varepsilon, \end{cases}$$

  for a fixed small threshold $\varepsilon > 0$.

- *Nesting function $\mathcal{N}_1$ the standard step index n.*
- *Resonance map $\mathcal{R}_1$ measuring stability of orbits (omitted for brevity).*

The iterative rule producing the sequence $a^{(1)}$ is

$$a^{(1)}[n] := C_\varepsilon\big(F(a^{(1)}, n)\big),$$

with initial seeds $a^{(1)}[0] = 0,\ a^{(1)}[1] = 1$.

Definition 5 (Target structure $\mathcal{S}_2$). *Let $\mathcal{S}_2 = (A_2, \mathcal{O}_2, \mathcal{N}_2, \mathcal{R}_2)$ be the UNNS-structure with:*

- $A_2 = \{\, 0 + 0i,\ 1 + 0i \,\}$ *(Gaussian seeds),*
- *Primary recursion operator $\widetilde{F} : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}$ given by complex addition*

$$\widetilde{F}(b, n) \;=\; b[n-1] + b[n-2],$$

  acting componentwise on complex values,

- *Gaussian inlaying (projection) operator $G : \mathbb{C} \to \mathbb{Z}[i]$ defined by*

$$G(z) = \mathrm{round}_{\mathbb{Z}[i]}(z) \;=\; \big(\mathrm{round}(\Re z)\big) + i\big(\mathrm{round}(\Im z)\big),$$

  i.e. rounding real and imaginary parts to nearest integers,

- *$\mathcal{N}_2, \mathcal{R}_2$ analogous to $\mathcal{S}_1$.*

The iterative rule in $\mathcal{S}_2$ is

$$b[n] := G\big(\widetilde{F}(b,n)\big),$$

with initial seeds $b[0] = 0 + 0i,\ b[1] = 1 + 0i$.

The embedding map ϕ

Define $\phi : A_1 \hookrightarrow A_2$ on the seeds by the natural real embedding

$$\phi(0) = 0 + 0i, \qquad \phi(1) = 1 + 0i,$$

and extend ϕ to finite sequences termwise:

$$\phi\big((a[0], a[1], \dots, a[n])\big) \;=\; \big(\phi(a[0]), \phi(a[1]), \dots, \phi(a[n])\big).$$

We regard ϕ as the candidate morphism $\mathcal{S}_1 \to \mathcal{S}_2$. Intuitively, ϕ embeds real sequence values into the Gaussian lattice via the real axis.

Operator-commutation up to bounded rounding

The exact commutation $\phi \circ C_\varepsilon \circ F = G \circ \widetilde{F} \circ \phi$ cannot hold in general because G performs rounding. We prove a controlled error statement showing the two compositions differ by at most a half-integer in the real and imaginary parts.

Proposition 2. *Let $a^{(1)}$ be the sequence generated by $\mathcal{S}_1$ and b the sequence generated by $\mathcal{S}_2$ with $b[0] = \phi(a^{(1)}[0])$, $b[1] = \phi(a^{(1)}[1])$. For every step $n \geq 2$,*

$$\Big|\Re\Big(\phi\big(C_\varepsilon(F(a^{(1)}, n))\big)\Big) \;-\; \Re\Big(G\big(\widetilde{F}(\phi(a^{(1)}), n)\big)\Big)\Big| \;\leq\; \tfrac{1}{2},$$

and similarly for the imaginary parts (which here are zero). Consequently

$$\Big\|\phi\big(C_\varepsilon(F(a^{(1)}, n))\big) - G\big(\widetilde{F}(\phi(a^{(1)}), n)\big)\Big\|_\infty \leq \tfrac{1}{2}.$$

Proof (sketch). Fix $n \geq 2$. Write $x := F(a^{(1)}, n) = a^{(1)}[n-1] + a^{(1)}[n-2] \in \mathbb{R}$. Two cases:

Case 1: $|x| < \varepsilon$. Then $C_\varepsilon(x) = 0$, hence $\phi(C_\varepsilon(x)) = 0 + 0i$. On the other hand, working in $\mathcal{S}_2$,

$$\widetilde{F}(\phi(a^{(1)}), n) = \phi(a^{(1)}[n-1]) + \phi(a^{(1)}[n-2]) = x + 0i.$$

Applying the Gaussian rounder G yields $G(x + 0i) = \mathrm{round}(x) + i \cdot 0$. Since $|x| < \varepsilon$, if we pick $\varepsilon \leq \frac{1}{2}$ then $|\,\mathrm{round}(x)| \leq \frac{1}{2}$. Thus $|\Re(0) - \mathrm{round}(x)| \leq \frac{1}{2}$, giving the desired bound.

Case 2: $|x| \geq \varepsilon$. Then $C_\varepsilon(x) = x$ and $\phi(C_\varepsilon(x)) = x + 0i$. Also $\widetilde{F}(\phi(a^{(1)}), n) = x + 0i$, and $G(x + 0i) = \mathrm{round}(x) + i \cdot 0$. The difference $|x - \mathrm{round}(x)| \leq \frac{1}{2}$ by always rounding to nearest integer. Hence the bound follows.

Combining the two cases yields the uniform $\frac{1}{2}$ bound on the real part. The imaginary component is identically zero in this embedding, so the ∞-norm bound follows. $\square$

The constant $\frac{1}{2}$ is sharp for nearest-integer rounding. One can reduce the bound by using different projection conventions (e.g. stochastic rounding) or by scaling the target lattice (embedding into $\lambda\mathbb{Z}[i]$).

Transport of attractors

The classical Fibonacci sequence grows like $\varphi^n/\sqrt{5}$ where $\varphi = (1+\sqrt{5})/2$ is the golden ratio. The source attractor is the projective direction of the growth (i.e. ratios $a[n+1]/a[n] \to \varphi$). Under ϕ the sequence is embedded along the real axis and G maps large real values to nearby Gaussian integers. Proposition **??** implies that for large n the lattice points $b[n] \in \mathbb{Z}[i]$ approximate $\varphi^n/\sqrt{5}$ to within an additive error $\leq \frac{1}{2}$. Therefore the growth direction (the attractor) is preserved in the sense that the *projective* limit (direction of growth) maps to the same real axis direction in the target lattice.

Formally, if we consider normalized projective states

$$\hat{a}[n] := \frac{a^{(1)}[n]}{a^{(1)}[n-1]} \qquad \text{and} \qquad \hat{b}[n] := \frac{\Re b[n]}{\Re b[n-1]},$$

then $\hat{a}[n] \to \varphi$ and $\hat{b}[n] \to \varphi$ as $n \to \infty$, up to errors vanishing relative to magnitude (the additive $\frac{1}{2}$ becomes negligible compared to φ^n). Thus the attractor is transported. □

Interpretation in the UNNS Topos

The above shows $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ is an operator-preserving morphism in the approximate sense required for realistic computational operators in UNNS: operators commute after embedding up to controlled projection errors. In categorical terms this is acceptable: morphisms in **UNNS** may be taken to preserve operator action modulo a bounded *repair error*, and such bounded errors are internalized by the resonance map $\mathcal{R}$.

Consequently:

- ϕ maps stable growth directions (the golden ratio attractor) of $\mathcal{S}_1$ to corresponding growth directions in $\mathcal{S}_2$.
- Collapse in $\mathcal{S}_1$ corresponds to (small) lattice rounding in $\mathcal{S}_2$; both serve to remove small-scale noise, and both preserve the large-scale attractor.

Extensions and remarks

1. One may strengthen the approximation by embedding into a dilated lattice $\lambda\mathbb{Z}[i]$ with $\lambda \gg 1$. Then rounding error becomes $\leq \frac{1}{2}$ in scaled units, corresponding to an absolute error of $\frac{1}{2}\lambda$ in original units but a relative error that vanishes for large sequence terms.
2. More generally, morphisms between UNNS structures can be required to satisfy $\phi \circ O_1 = O_2 \circ \phi$ exactly when the operators are algebraic and compatible (e.g. linear operators and linear embeddings). For nonlinear operators (like rounding), the approximate commutation above is a natural and useful relaxation.

3. The same construction works if we replace $\mathbb{Z}[i]$ with $\mathbb{Z}[\omega]$ (Eisenstein integers), with an analogous rounding bound.

Conclusion. This worked example demonstrates how a concrete morphism in the UNNS Topos can be constructed, how operator action commutes up to bounded projection error, and how attractors (growth directions) are transported through the morphism. It provides a useful template for further, more intricate examples involving composite operators and higher-dimensional lattice embeddings.

6.1 Interpretation

In classical topos theory, truth is carried by $\Omega = \{0, 1\}$. In UNNS, truth is replaced by stability: a sequence is "true" if it stabilizes, "false" if it diverges. Collapse functions as the subobject classifier.

7 Diagrammatic Comparison

8 Philosophical Reflection

- Classical: structure = static interpretation.
- UNNS: structure = recursive attractor.
- Category of UNNS = logic as dynamics.

9 Conclusion

UNNS enriches the concept of structure. Instead of static interpretations, we obtain recursive attractors, resonance fields, and categorical dynamics. The UNNS Topos frames recursion as logic, with collapse as truth, stabilization as validity, and resonance as consistency. This suggests UNNS could provide a foundation for a dynamic, recursion-based mathematics.